TWO-POPULATION REPLICATOR DYNAMICS OF DENSITY FUNCTIONS IN LINEAR QUADRATIC GAMES

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CERTIFICATION OF APPROVAL

I certify that I have read *Two-Population Replicator Dynamics of Density Functions in Linear Quadratic Games* by Amy Sue Morrow, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University

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Standard evolutionary models in game theory use the replicator dynamics inspired by predator-prey models. Describing populations by densities allows the extension of the replicator dynamics to a continuum of pure decision choices as is desired in many games such as product pricing in economics. In this paper, we extend results found in [5] to show that in the two-population case, the densities each converge to a peak. We then determine the cases which will ensure long term convergence of this peak to the interior Nash equilibrium in the linear-quadratic case.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

PREFACE AND/OR ACKNOWLEDGEMENTS

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Introduction

Game theory models competitive situations where there are two or more decision makers, each with a well-defined set of decision choices, his strategies. The decision makers are called players. In noncooperative game theory, the players each act in their own best interest. Each player has a hierarchy of preferences for outcomes that is usually expressed as a utility/payoff function. Players are typically assumed to be rational, meaning that each player wishes to do the best for himself, assuming others behave the same. Game theory seeks to determine the best strategy.

In economics, for example, when a small group of firms are the only firms in a market, they form a game called an oligopoly. The firms must decide either at what price to sell their object, or in what quantity to produce it, in order to maximize their profits. As early as 1838, Augustin Cournot developed the special case of a duopoly, an oligopoly with only two firms [7]. The Cournot duopoly model, applicable when firms produce identical products, has the selling price of a product determined by demand, and the two firms each try to maximize their profits by adjusting the quantity they produce. If, however, the firms are producing differentiated products, such as Coke and Pepsi, the Bertrand model, developed by Joseph Bertrand in 1883 as a response to Cournot, applies, and the firms set their prices with demand then determining the quantity produced.

From the game theoretic standpoint, both of these models fall under the category of noncooperative, continuous games. They are first considered games because players (firms) have decision choices to make (quantity/price), and they are interested in choosing a strategy that would optimize outcomes for themselves based on their personal hierarchy of preferences (more money is better). Their actions are noncooperative because the firms seek their own profit maximizing strategies over what would be good for the group as a whole. At each iteration of time, the firms get to choose their price or quantity. Finally, these games are classified continuous as their strategies come from a continuum of choices.

Regardless of which type of game we discuss, the heart of game theory strives to find an equilibrium–a set of strategies for each player such that no other strategy gives a higher payoff given the decision choices of the other players. Between 1950 and 1953, John Nash produced works defining the Nash Equilibrium concept. Formally, if we are given a set of n players, each with a finite set of pure strategies to choose from, then an ordered n-tuple $(x_1, \ldots, x_n)$ is a strategy profile, where each player i has the strategy x_i. The strategy profile $(x_1, \ldots, x_n)$ is then a Nash equilibrium if for each player i, no strategy other than x_i would yield a higher payoff given the others. In his seminal work, Nash showed that every game when played only once (rather than repeated through time) has at least one Nash equilibrium if one allows probabilistic play.

This naturally leads to questions surrounding what happens to the Nash equilibria as the game is played repeatedly, or continuously, through time. In order to begin studying this, we consider a large population of players each programmed to a strategy. At a given instance, two players are drawn at random from the population to play against each other. This approach, known as “evolutionary game theory” takes two major routes: examining effects of mutations and examing a selection mechanism. Examining the criterion of evolutionary stability involves a population’s reaction to mutations, while the replicator dynamics instead regards how the population selects favorable strategies.

To examine evolutionarily stable strategies (ESS), consider that each member of the large population is programmed to play a particular strategy, with the bulk of the population playing the exact same strategy repeatedly. One can then evaluate which strategies are evolutionarily stable by mutating a small percentage of the population to play a different strategy, and seeing how the main population performs. A strategy is ESS if it performs better than the mutated fraction, presuming the mutated fraction is not too large. These strategies form a subset of the Nash equilibria.

The other approach to evolutionary game theory examines the selection mechanism via the replicator dynamics. An initial mix of the population is given with portions of players programmed to play different pure strategies. In the traditional development, players may only choose from a finite set of pure decision choices. Favorable strategies are those that perform better than the group as a whole is faring. The replicator dynamics selects favorable strategies by breeding more players playing those strategies; players playing underperforming strategies die off.

There are, however, many models whose games involve infinitely many (perhaps a continuum of) pure strategy choices. For instance, if we adopt Bertrand’s model for a duopoly, we are then looking at two firms that can choose the price at which they sell their good with quantity determined by demand. The firms have an infinite, rather than discrete, number of choices for the price of their good. In order to discuss the actions of these firms in the long run, it becomes necessary to discuss the replicator dynamics for these games in terms of the continuum. Extension of the replicator dynamics to the continuum is straightforward.

Most of those extending the replicator dynamics to a continuum of pure decision choices examine the stability of an ESS under the replicator dynamics. When strategies are taken from a finite set, an ESS is asymptotically stable under the replicator dynamics. However, in [6] Oechssler and Riedel have shown that when there are infinitely many pure strategies, an ESS may not remain stable. Even requiring the space from where the strategies are taken to be compact and the payoff functions to be continuous is not enough to guarantee the stability of a strict Nash equilibrium.

Eshel et al. in [1] demonstrate that the concept of Continuously Stable Strategies (CSS) is sufficient to ensure long term convergence of that strategy. A CSS is an ESS, where if the population as a whole deviates some sufficiently small amount from the ESS, then it is the mutations closer to the ESS which survive. Eshel and Sansone argue in [2] that when considering the extension of pure decision choices to the continuum, one needs to take into consideration the topology over which you measure the population. They introduce the concept of a Continuously Replicator Stable Strategy (CRSS), and show that this is necessary and sufficient for asymptotic stability of the replicator dynamics in the weak topology. Under the maximal shift topology, however, CRSS is only an almost sufficient condition for stability under the replicator dynamics. This condition, along with CSS, are implied by the criterion of Evolutionary Robustness (ER).

Using density functions to describe the population, Langlois has examined the same problem of long term stability under the replicator dynamics with pure strategies taken from the continuum [5]. Langlois has shown that, given an initial density function describing the population with a game concave in its decision choices, the solution of the replicator dynamics is a density that approaches a Dirac mass as time approaches infinity. Furthermore, under assumptions that imply the existence of a unique symmetric Nash equilibrium that is interior, he has demonstrated that the solution to the replicator dynamics approaches the Dirac delta mass at the Nash equilibrium, δ_N (as time approaches infinity) [5].

In this paper, we extend Langlois' approach to the replicator dynamics of the two-population game where players with different payoff structures play against each other. We consider the concave linear quadratic case, which is the case that models the Cournot and Bertrand duopolies as described above. We determine the necessary conditions for the replicator dynamics for each population to converge in the long run to the interior Nash equilibrium. Additionally, we see under which conditions we can guarantee an unstable Nash equilibrium. The results are then numerically verified in the final section.

The Replicator Dynamics

We are given an initial mix of the population, with a certain percentage of the population each playing a different strategy. The classic development only allows the players to choose from the same finite set $\{x_1, \ldots, x_k\}$ of pure decision choices, and a population state $\mu = (\mu_1, \ldots, \mu_k)$ would be defined where each μ_i is the share of the population playing x_i. If we let $P^*(x_i, \mu) = \sum_{j=1}^{k} P(x_i, x_j)\mu_j$ be the payoff for playing x_i against the population μ where P is the payoff against a single player, then the average population payoff, also known as the "population fitness", would be $F(\mu) = \sum_{i=1}^{k} \mu_i P^*(x_i, \mu)$.

We now let $N(t)$ be the number of players in the population, and we let $n_i(t)$ denote the number of players playing strategy x_i. Then we have $n_i(t) = \mu_i(t)N(t)$. We can now impose a reproductive structure on the number of people playing x_i by letting $P^*(x_i, \mu)$ be the number of players "born" programmed to play x_i from each individual already playing x_i. If we let $\beta \geq 0$ and $\delta \geq 0$ be the ambient birth and death rates, respectively, we then have the following population dynamics:

$$\frac{dn_i(t)}{dt} = (\beta + P^*(x_i, \mu) - \delta)n_i(t).$$

Differentiating the identity $N(t)\mu_i(t) = n_i(t)$ yields:

$$N(t)\frac{d\mu_i(t)}{dt} = \frac{dn_i(t)}{dt} - \frac{dN(t)}{dt}\mu_i(t).$$

This can be simplified to yield the replicator dynamics:

$$\frac{d\mu_i}{dt} = [P^*(x_i, \mu) - F(\mu)]\mu_i$$

[10]. As desired, strategies that perform better than the average population breed more players playing that strategy, whereas strategies that perform worse die off.

Extension of the replicator dynamics to the continuum is straightforward. We take μ to be a density function $\mu(x)$ where $\int_a^b \mu(x)dx$ is the probability that a random player plays x in $[a, b]$ with payoff $P(x, y)$ against player y. We can now consider the payoff, $P^*(x|\mu) = \int_0^1 P(x, y)\mu(y)dy$, to the player playing x against the rest of the population μ and the average population performance, or population fitness, $F(\mu) = \int_0^1 P^*(x|\mu)\mu(x)dx$.

At this point, the extension of the replicator dynamics to the continuum is apparent, yielding the differential equation

$$\frac{d\mu_t(x)}{dt} = \mu_t(x)\left(P^*(x|\mu_t) - F(\mu_t)\right).$$

If the payoff for playing x is greater than the average population performance, $\mu_t(x)$ will increase with time, meaning that more players will choose to play x. Similarly, if x performs worse than the average population, $\mu_t(x)$ will decrease. If we are given an initial funtion $\mu_0(x) \in \mathcal{D}$ (defined in Extending Existing Results) for our population at time $t = 0$, then $\mu_t \in \mathcal{D}$ as well, and this equation for $\mu_t'(x)$ has a unique solution and can be implicitly solved for $\mu_t(x)$ to get

$$\mu_t(x) = \mu_0(x)e^{\int_0^t (P^*(x|\mu_\tau) - F(\mu_\tau))d\tau},$$

which is defined for all t since P^* and F are continuous and bounded.

The Two-Population Model

We wish to evaluate what becomes of a two player game with players chosen from different populations whose pure strategy choices may be picked from $[0,1]$ but whose payoff structures may differ. We consider two populations, one described by the density function μ and the other by the density function λ. The payoff for a player from the μ population playing pure strategy x against a single player playing y from the λ population is $P(x,y)$, and the payoff when playing x against the λ population is $P^*(x|\lambda)$. Similarly, $Q(x,y)$ will be the payoff for a player from the λ population, playing y, against a player playing x, and $Q^*(y|\mu)$ will denote the payoff of playing y against the μ population. Given $P(x,y)$, we can consider $P^*(x|\lambda_t)$ as the expected payoff for playing x. We then have

$$P^*(x|\lambda_t) = \int_0^1 P(x,y)\lambda_t(y)dy,$$

and we similarly have

$$Q^*(y|\mu_t) = \int_0^1 Q(x,y)\mu_t(x)dx.$$

The fitness of the population μ_t is a measure of how the population is performing and is given by

$$F(\mu_t|\lambda_t) = \int_0^1 P^*(x|\lambda_t)\mu_t(x)dx,$$

and the fitness of the population λ_t is

$$G(\lambda_t|\mu_t) = \int_0^1 Q^*(y|\mu_t)\lambda_t(y)dy.$$

From these equations, we generalize the replicator dynamics to the two-population case with the system of differential equations given by

$$\frac{\partial \mu_t(x)}{\partial t} = \mu_t(x)(P^*(x|\lambda_t) - F(\mu_t|\lambda_t))$$

and

$$\frac{\partial \lambda_t(y)}{\partial t} = \lambda_t(y)(Q^*(y|\mu_t) - G(\lambda_t|\mu_t)).$$

Given initial densities μ_0 and λ_0, this system can be solved implicitly to yield

$$\mu_t(x) = \mu_0(x)e^{\int_0^t (P^*(x|\lambda_\tau) - F(\mu_\tau|\lambda_\tau))d\tau} \tag{1}$$

and

$$\lambda_t(y) = \lambda_0(y)e^{\int_0^t (Q^*(y|\mu_\tau) - G(\lambda_\tau|\mu_\tau))d\tau}.$$

The solution for μ_t is valid for all $t \geq 0$ since $P^*(x|\lambda_\tau) - F(\mu_\tau|\lambda_\tau)$ is continuous in τ and bounded for all x, λ_τ, and μ_τ. Similarly, our solution for λ_t is defined for all $t \geq 0$.

Extending Existing Results

It is known in the single-population case that under the replicator dynamics with assumptions stated in the introduction, the population density converges to the Dirac delta mass at the unique interior Nash equilibrium [5]. We wish to generalize this result. We proceed by first adapting the proof in order to obtain a similar concentration of mass result for the two population case and to also obtain the system of differential equations found in Theorem 1 describing the behavior of the peaks of our populations.

We adopt the following notation from [5]: Let $\mathcal{D} = \{\mu \in C^2[0,1] : \forall x, \mu(x) > 0,\ \int_0^1 \mu(x)dx = 1\}$ and $I(\xi, \varepsilon) = [0,1] \cap [\xi - \varepsilon, \xi + \varepsilon]$. We make the assumption that both $P(x,y)$ and $Q(x,y)$ are concave in their respective decision maker's variable, so we insist $\frac{\partial^2 P}{\partial x^2}, \frac{\partial^2 Q}{\partial y^2} < 0$. We also assume that $\left.\frac{\partial P}{\partial x}\right|_{(0,y)}, \left.\frac{\partial Q}{\partial y}\right|_{(x,0)} > 0$ and $\left.\frac{\partial P}{\partial x}\right|_{(1,y)}, \left.\frac{\partial Q}{\partial y}\right|_{(x,1)} < 0$, in order to ensure our dynamics remain in $[0,1] \times [0,1]$. Additionally, we call $\xi(t)$ a peak, meaning $\xi(t)$ is a value of x such that $\mu_t(x)$ is maximum, and similarly $\eta(t)$ is a peak means $\eta(t)$ is a value of y such that $\lambda_t(y)$ is maximum.

Theorem 1 ***(Peak Behavior Theorem.)*** *Under the above assumptions with μ_0, $\lambda_0 \in \mathcal{D}$, there exists T such that for all $t \geq T$, there exists $\xi(t)$ and $\eta(t)$ where $\xi(t)$ and $\eta(t)$ are the unique interior peaks of μ_t and λ_t respectively; moreover,*

$$\frac{d\xi(t)}{dt} = \frac{-\frac{\partial P^*}{\partial x}(\xi|\lambda_t)}{\left(\frac{\partial}{\partial x}\left(\frac{1}{\mu_0}\frac{\partial \mu_0}{\partial x}\right) + \int_0^t \frac{\partial^2 P^*}{\partial x^2}(x|\lambda_\tau)d\tau\right)_{x=\xi}} \tag{2}$$

$$\frac{d\eta(t)}{dt} = \frac{-\frac{\partial Q^*}{\partial y}(\eta|\mu_t)}{\left(\frac{\partial}{\partial y}\left(\frac{1}{\lambda_0}\frac{\partial \lambda_0}{\partial y}\right) + \int_0^t \frac{\partial^2 Q^*}{\partial y^2}(y|\mu_\tau)d\tau\right)_{y=\eta}}$$

(generalizing Lemma 6 from [5]).

We will prove this result by rephrasing each of the lemmata found in [5] leading up to this result, and by indicating necessary adjustments to the proofs. We will conclude this section by providing a brief synopsis deriving these equations.

Lemma 1 *Assume $\mu_0, \lambda_0 \in \mathcal{D}$. Then there exists constants $\underline{a_\mu}$, $\underline{a_\lambda}$, $\overline{a_\mu}$, $\overline{a_\lambda}$, $\overline{b_\mu} > \underline{b_\mu} > 0$, $\overline{b_\lambda} > \underline{b_\lambda} > 0$, such that for all $t \geq 0$,*

$$L_\mu(t) = \underline{a_\mu} + \underline{b_\mu} t \leq -\frac{\partial}{\partial x}\left(\frac{1}{\mu_t}\frac{\partial \mu_t}{\partial x}\right) \leq \overline{a_\mu} + \overline{b_\mu} t = M_\mu(t)$$

and

$$L_\lambda(t) = \underline{a_\lambda} + \underline{b_\lambda} t \leq -\frac{\partial}{\partial y}\left(\frac{1}{\lambda_t}\frac{\partial \lambda_t}{\partial y}\right) \leq \overline{a_\lambda} + \overline{b_\lambda} t = M_\lambda(t).$$

Proof: The proof of this statement is similar to that of Lemma 1 in [5]. By taking the implicit solution to the replicator dynamics, the same proof applies by differentiating twice with respect to the decision maker's variable since F and G do not depend on x or y and since each payoff function, P and Q, is concave in the decision maker's variable. $\square$

Therefore, the two corollaries from Lemma 1 in [5]. We have

$$\lim_{t\to\infty} \frac{M_\mu}{L_\mu} = \overline{b_\mu}/\underline{b_\mu} = r_\mu \quad \text{and} \lim_{t\to\infty} \frac{M_\lambda}{L_\lambda} = \overline{b_\lambda}/\underline{b_\lambda} = r_\lambda.$$

Additionally, we have $\lim_{t\to\infty} \frac{\partial}{\partial x}\left(\frac{1}{\mu_t}\frac{\partial \mu_t}{\partial x}\right) = \lim_{t\to\infty} \frac{\partial}{\partial y}\left(\frac{1}{\lambda_t}\frac{\partial \lambda_t}{\partial y}\right) = -\infty$.

Since the proofs of the next three lemmata only depend on μ and λ being densities, we can apply the results without altering the proofs.

Lemma 2 *Let $\mu_t, \lambda_t \in \mathcal{D}$ and suppose there exist $L_\mu, M_\mu, L_\lambda, M_\lambda$ such that*

$$L_\mu \leq -\frac{\partial}{\partial x}\left(\frac{1}{\mu_t}\frac{\partial \mu_t}{\partial x}\right) \leq M_\mu \text{ and } L_\lambda \leq -\frac{\partial}{\partial y}\left(\frac{1}{\lambda_t}\frac{\partial \lambda_t}{\partial y}\right) \leq M_\lambda.$$

Then for any $\xi, \eta \in [0,1]$, we have

$$\mu_t(\xi)e^{-\psi(x-\xi)-\frac{M_\mu}{2}(x-\xi)^2} \leq \mu_t(x) \leq \mu_t(\xi)e^{-\psi(x-\xi)-\frac{L_\mu}{2}(x-\xi)^2}$$

and

$$\lambda_t(\eta)e^{-\phi(y-\eta)-\frac{M_\lambda}{2}(y-\eta)^2} \leq \lambda_t(y) \leq \lambda_t(\eta)e^{-\phi(y-\eta)-\frac{L_\lambda}{2}(y-\eta)^2}$$

where $\psi = \left(-\frac{1}{\mu_t}\frac{\partial \mu_t}{\partial x}\right)\Big|_{x=\xi}$ and $\phi = \left(-\frac{1}{\lambda_t}\frac{\partial \lambda_t}{\partial y}\right)\Big|_{y=\eta}$

Lemma 3 *Under the assumptions of Lemma 2, if $M_\mu, M_\lambda \geq 2$ then $\mu_t(x) \leq (2\psi + M_\mu)e^{-\psi(x-\xi)-\frac{L_\mu}{2}(x-\xi)^2}$ and $\lambda_t(x) \leq (2\phi + M_\lambda)e^{-\phi(y-\eta)-\frac{L_\lambda}{2}(y-\eta)^2}$ where ξ and η are peaks of μ_t and λ_t respectively.*

Lemma 4 *Given $\delta > 0$. Let ξ, η be the peaks of μ, λ respectively. Then, under the assumptions of Lemma 2, for any $0 < \varepsilon < \delta$, there exists L_0 such that for all $L_\mu, L_\lambda \geq L_0$ and M_μ and M_λ such that $\delta L_\mu \leq M_\mu \leq 2\delta L_\mu$ and $\delta L_\lambda \leq M_\lambda \leq 2\delta L_\lambda$, we have $\int_{I(\xi,\varepsilon)} \mu(x)dx \geq 1-\varepsilon$ and $\int_{I(\eta,\varepsilon)} \lambda(x)dx \geq 1-\varepsilon$.*

Since we are able to use each of Lemmas 1 through 4, we obtain as a consequence the Two-Population Mass Concentration Theorem.

Theorem 2 ***(Two-Population Mass Concentration Theorem.)*** *Given the conditions on our game, for any initial densities $\lambda_0, \mu_0 \in \mathcal{D}$ and any $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geq T$,*

$$\int_{I(\xi(t),\varepsilon)} \mu_t(x)dx \geq 1 - \varepsilon \quad \textit{and} \quad \int_{I(\eta(t),\varepsilon)} \lambda_t(x)dx \geq 1 - \varepsilon$$

where $\xi(t)$ and $\eta(t)$ are the peaks of $\mu_t(x)$ and $\lambda_t(x)$ respectively.

The proof for this next lemma also directly applies to the two-population case.

Lemma 5 *Let $\alpha(\xi, \lambda) = \frac{\partial P^*}{\partial x}(\xi|\lambda) - \frac{\partial P}{\partial x}(\xi, \eta)$ and $\beta(\eta, \mu) = \frac{\partial Q^*}{\partial y}(\eta|\mu) - \frac{\partial Q}{\partial y}(\eta, \xi)$. Also let $A = \max_{x\in[0,1]} \left|\frac{\partial^2 P}{\partial x \partial y}\right| - \max_{x\in[0,1]} \left|\frac{\partial P}{\partial x}\right|$ and $B = \max_{x\in[0,1]} \left|\frac{\partial^2 Q}{\partial y \partial x}\right| - \max_{x\in[0,1]} \left|\frac{\partial Q}{\partial y}\right|$. Then for any $\varepsilon > 0$, if Lemma 4 holds, then $|\alpha(\xi, \lambda)| < A\varepsilon$ and $|\beta(\eta, \mu)| < B\varepsilon$.*

Synopsis of Proof of The Peak Behavior Theorem: By applying the proof to Lemma 6 in [5], one can show $\frac{\partial \mu_t(x)}{\partial x} = \frac{\partial \lambda_t(y)}{\partial y} = 0$ at an interior point $(x = \xi, y = \eta)$. From (1), we obtain

$$\ln \mu_t(x) = \ln \mu_0(x) + \int_0^t (P^*(x|\lambda_\tau) - F(\mu_\tau|\lambda_\tau))d\tau$$

and similarly for $\lambda_t(y)$. Differentiating this equation yields

$$\frac{1}{\mu_t(x)} \frac{\partial \mu_t(x)}{dx} = \frac{1}{\mu_0(x)} \frac{\partial \mu_0(x)}{dx} + \int_0^t \frac{\partial (P^*(x|\lambda_\tau)}{\partial x} d\tau$$

and similarly for $\lambda_t(y)$. Therefore, we can conclude $(\xi(t), \eta(t))$ gives a solution to the system

$$\begin{aligned}\frac{1}{\mu_t(x)}\frac{\partial\mu_t(x)}{dx} &= \frac{1}{\mu_0(x)}\frac{\partial\mu_0(x)}{dx} + \int_0^t \frac{\partial(P^*(x|\lambda_\tau)}{\partial x}d\tau = 0\\ \frac{1}{\lambda_t(y)}\frac{\partial\lambda_t(y)}{dy} &= \frac{1}{\lambda_0(y)}\frac{\partial\lambda_0(y)}{dy} + \int_0^t \frac{\partial(Q^*(y|\mu_\tau)}{\partial y}d\tau = 0\end{aligned}$$

By the Implicit Function Theorem, there exists $x = \xi(t)$, $y = \eta(t)$, satisfying the given equations for $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$. Differentiating these equations with respect to t, evaluating at $x = \xi(t), y = \eta(t)$, and solving the resulting equations for $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$ will produce the desired equations. □

Two-Population Theory for Linear Quadratic Games

In terms of Bertrand's model for a duopoly, the firms have an infinite, rather than discrete, number of choices for the price of their good. If we let x be the price for Firm X and y be the price for Firm Y, then a possible model for the quantity sold by X would be $q_X = a - bx + cy$ where $a, b, c > 0$ so that the quantity sold by X is a decreasing function of the firm's price but an increasing function of Firm Y's price. The payoff to Firm X would then be $\pi_x = p_x \cdot q_x = x(a - bx + cy)$. We now focus on this linear-quadratic case as our methods allow us to make a successful analysis of the long term behavior of the populations. Therefore, we take $P(x, y) = x(a - bx + cy)$ and $Q(x, y) = y(d - ey + fx)$.

We would like to use the results of the Peak Behavior Theorem. In order to meet the concavity assumption in the linear quadratic case, we must have that $b, e > 0$. Furthermore, to meet the assumptions about the boundary, we must require that $\frac{\partial P}{\partial x}\big|_{(0,y)} > 0$ and $\frac{\partial P}{\partial x}\big|_{(1,y)} < 0$ for all $y \in [0, 1]$. Therefore we must have $a + cy > 0$ and $a - 2b + cy < 0$ for all $y \in [0, 1]$. These lead to the assumptions that $0 < a < 2b$ and $0 < a + c < 2b$. Similarly, we can deduce that we must have $0 < d < 2e$ and $0 < d + f < 2e$. This establishes the conditions our payoff functions must have in order to use the results of the Peak Behavior Theorem to obtain stability results for ξ and η.

We can now fill in the equations from the Peak Behavior Theorem in the linear quadratic case by plugging in the necessary quantities. First, we note that $\frac{\partial P}{\partial x} =$

$a - 2bx + cy$ and $\frac{\partial^2 P}{\partial x^2} = -2b$. We can now calculate $\frac{\partial P^*(\xi|\lambda_t)}{\partial x}$:

$$\begin{aligned}\frac{\partial P^*(\xi|\lambda_t)}{\partial x} &= \frac{\partial}{\partial x}\left(\int_0^1 P(x,y)\lambda_t(y)dy\right)\Big|_{x=\xi} \\ &= \int_0^1 \frac{\partial P}{\partial x}(x,y)\lambda_t(y)dy\Big|_{x=\xi} \\ &= \int_0^1 (a - 2bx + cy)\lambda_t(y)dy\Big|_{x=\xi} \\ &= a - 2b\xi + c\int_0^1 y\lambda_t(y)dy.\end{aligned}$$

But $EY = \int_0^1 y\lambda_t(y)dy$ is the expectation of the continuous random variable, Y, with density function λ_t. So we get $\frac{\partial P^*(\xi|\lambda_t)}{\partial x} = a - 2b\xi + cEY$, and similarly $\frac{\partial Q^*(\eta|\mu_t)}{\partial y} = d - 2e\eta + fEX$ where EX is the expectation of the random variable X associated with μ_t. Simple calculations will also show $\int_0^t \frac{\partial^2 P^*(\xi|\lambda_\tau)}{\partial x^2} d\tau\Big|_{x=\xi} = -2bt$ and $\int_0^t \frac{\partial^2 Q^*(\eta|\mu_\tau)}{\partial y^2} d\tau\Big|_{y=\eta} = -2et$. We also have the quantities $\frac{\partial}{\partial x}\left(\frac{1}{\mu_0} \cdot \frac{\partial \mu_0}{\partial x}\right)\Big|_{x=\xi} = K_\mu(\xi)$, a function of ξ, and $\frac{\partial}{\partial y}\left(\frac{1}{\lambda_0} \cdot \frac{\partial \lambda_0}{\partial y}\right)\Big|_{y=\eta} = K_\lambda(\eta)$, a function of η.

Under the conditions we imposed together with the assumption $\mu_0, \lambda_0 \in \mathcal{D}$, the Peak Behavior Theorem implies that there exists a T such that for all $t \geq T$,

$$\frac{d\xi(t)}{dt} = -\frac{a - 2b\xi + cEY}{K_\mu(\xi) - 2bt} \qquad \frac{d\eta(t)}{dt} = -\frac{d - 2e\eta + fEX}{K_\lambda(\eta) - 2et} \tag{3}$$

where $\xi(t)$ is the peak of μ_t and $\eta(t)$ is the peak of λ_t. We now begin to show that this system converges to a system on which we can more easily evaluate the stability of solutions.

Lemma 6 *Given any $\varepsilon > 0$, there exists T_1 such that for all $t \geq T_1$, $\left|\frac{1}{t}K_\mu(\xi)\right| < \varepsilon$.*

Proof: Since $\mu_0 \in \mathcal{D}$, $K_\mu(\xi) = \frac{\partial}{\partial x}\left(\frac{1}{\mu_0} \cdot \frac{\partial \mu_0}{\partial x}\right)\Big|_{x=\xi}$ is continuous on the compact interval $[0,1]$, and hence there exists $M \geq 0$ such that $|K_\mu(\xi)| \leq M$ for all $\xi \in [0,1]$. Therefore, taking $T_1 = \frac{M}{\varepsilon}$, for $t \geq T_1$ we have $\left|\frac{1}{t}K_\mu(\xi)\right| \leq \left|\frac{1}{t}\right| M < \frac{\varepsilon}{M}M = \varepsilon$, as desired. □

Similarly, there exists T_1' such that for all $t \geq T_1'$, $\left|\frac{1}{t}K_\lambda(\eta)\right| < \varepsilon$ for any given $\varepsilon > 0$.

To further simplify our equations, we look at the quantities EX and EY. Recall, $EX = \int_0^1 x\mu_t(x)dx$, and $EY = \int_0^1 y\lambda_t(y)dy$.

Lemma 7 *Given $\delta > 0$, there exists T_2 such that for all $t \geq T_2$, $|EX - \xi| < \delta$.*

Proof: By the Two-Population Mass Concentration Theorem, for $0 < \varepsilon < \frac{\delta}{2}$, there exists T_2 such that for all $t \geq T_2$, $\int_{I(\xi,\varepsilon)} \mu_t(x)dx \geq 1-\varepsilon$. Therefore $\int_{[0,1]\setminus I(\xi,\varepsilon)} \mu_t(x)dx \leq \varepsilon$. But since $0 \leq x \leq 1$, we have

$$\int_{[0,1]\setminus I(\xi,\varepsilon)} x\mu_t(x)dx \leq \int_{[0,1]\setminus I(\xi,\varepsilon)} \mu_t(x)dx \leq \varepsilon.$$

From that,

$$\int_0^1 x\mu_t(x)dx \leq \int_{I(\xi,\varepsilon)} x\mu_t(x)dx + \varepsilon.$$

Using this and since $I(\xi,\varepsilon) \subseteq [0,1]$, we have

$$\int_{I(\xi,\varepsilon)} x\mu_t(x)dx \leq \int_0^1 x\mu_t(x)dx \leq \int_{I(\xi,\varepsilon)} x\mu_t(x)dx + \varepsilon.$$

But $\int_{I(\xi,\varepsilon)} x\mu_t(x)dx \geq (\xi-\varepsilon)\int_{I(\xi,\varepsilon)} \mu_t(x)dx \geq (\xi-\varepsilon)(1-\varepsilon)$. Also, $\int_{I(\xi,\varepsilon)} x\mu_t(x)dx+\varepsilon \leq (\xi+\varepsilon)\int_{I(\xi,\varepsilon)} \mu_t(x)dx+\varepsilon \leq (\xi+\varepsilon)\int_0^1 \mu_t(x)dx+\varepsilon = (\xi+\varepsilon)\cdot 1+\varepsilon$ since $\int_0^1 \mu_t(x)dx = 1$ as μ_t is a density. Therefore,

$$(\xi-\varepsilon)(1-\varepsilon) \leq \int_0^1 x\mu_t(x)dx \leq (\xi+\varepsilon)+\varepsilon.$$

Therefore,

$$\varepsilon^2 - \varepsilon(\xi+1) \leq EX - \xi \leq 2\varepsilon.$$

If $0 \leq \varepsilon^2 - \varepsilon(\xi+1)$, then $|EX-\xi| \leq 2\varepsilon$. If $0 > \varepsilon^2 - \varepsilon(\xi+1)$, then

$$|\varepsilon^2 - \varepsilon(\xi+1)| = \varepsilon(\xi+1) - \varepsilon^2 = \varepsilon(\xi+1-\varepsilon) \leq 2\varepsilon$$

since $\xi \in [0,1]$, and therefore, $|EX-\xi| \leq 2\varepsilon$. Hence, for all $t \geq T_2$, $|EX-\xi| \leq 2\varepsilon < 2\frac{\delta}{2} = \delta$. □

A similar argument works for EY as well. Therefore, we can write (3) as

$$\frac{d\xi}{dt} = -\frac{1}{t}\cdot\frac{a-2b\xi+c(\eta+\phi_y(t))}{\frac{1}{t}K_\mu(\eta)-2b} \quad \text{and} \quad \frac{d\eta}{dt} = -\frac{1}{t}\cdot\frac{d-2e\eta+f(\xi+\phi_x(t))}{\frac{1}{t}K_\lambda(\xi)-2e}. \qquad (4)$$

where $\phi_y(t)$, $\phi_x(t)$, $\frac{1}{t}K_\lambda$, and $\frac{1}{t}K_\mu$ converge to 0 as t approaches infinity. We will now make a couple changes in variable to put our equations in a form we can better deal with.

Lemma 8 *If $cf \neq 4eb$, a Nash equilibrium for this game is given by $\xi_N = \frac{cd+2ea}{4eb-cf}$ and $\eta_N = \frac{2bd+af}{4eb-cf}$.*

Proof: If $\frac{\partial P}{\partial x} = 0$, $\frac{\partial Q}{\partial y} = 0$, we have a Nash equilibrium. Solving the resulting system of equations for ξ and η produces our desired result. □

We will place additional conditions on our coefficients to ensure that our Nash equilibrium falls within the interval $[0, 1]$. We insist that $0 \leq \eta_N, \xi_N \leq 1$.

We change variables to the Nash equilibrium by first letting $\xi' = \xi - \xi_N$ and $\eta' = \eta - \eta_N$ so that $\frac{d\xi'}{dt} = \frac{d\xi}{dt}$, $\frac{d\eta'}{dt} = \frac{d\eta}{dt}$, $\xi = \xi' + \xi_N$, and $\eta = \eta' + \eta_N$. This transforms our system (3) into

$$\frac{d\xi'}{dt} = \frac{1}{t} \cdot \frac{2b\xi' - c(\eta' + \phi_y(t))}{\frac{1}{t}K_\mu(\eta' + \eta_N) - 2b} \quad \text{and} \quad \frac{d\eta'}{dt} = \frac{1}{t} \cdot \frac{2e\eta' - f(\xi' + \phi_x(t))}{\frac{1}{t}K_\lambda(\xi' + \xi_N) - 2e}.$$

We can now get rid of the $\frac{1}{t}$ factor by changing variables to $s = \ln t$. This change yields the system

$$\frac{d\xi'}{ds} = \frac{2b\xi' - c(\eta' + \phi_y(e^s))}{e^{-s}K_\mu(\eta' + \eta_N) - 2b} \quad \text{and} \quad \frac{d\eta'}{ds} = \frac{2e\eta' - f(\xi' + \phi_x(e^s))}{e^{-s}K_\lambda(\xi' + \xi_N) - 2e}. \tag{5}$$

In order to gain some stability results on this system, we need the following lemma.

Lemma 9 *Suppose $\frac{dx}{dt} = Ax + G(x, t)$ is a system of 2 equations in 2 unknowns.*

Suppose $A = Q^{-1}\Lambda Q$, *where* $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ *where* λ_1, λ_2 *have negative real parts. Suppose also that* $G(x,t)$ *converges to* 0 *uniformly in* x *as* t *approaches infinity, where* G *is continuous and bounded for all* x *and* t. *Then* $x(t) = 0$ *is an asymptotically stable solution of* $\frac{dx}{dt} = Ax + G(x,t)$.

Proof: Let $Qx = y$ (which may take complex values), and multiply $\frac{dx}{dt} = Ax + G(x,t)$ by Q. We then have $\frac{dy}{dt} = \Lambda y + G(Q^{-1}y, t)$. Let $H(y,t) = G(Q^{-1}y,t)$, and note that H may also take on complex values as Q may have complex entries. Solving for y, we obtain $y(t) = e^{\Lambda(t-t_0)}y_0 + \int_{t_0}^t e^{\Lambda(t-s)}H(y(s),s)ds$. Let us take a look at the j-th entry of $y(t)$: $y_j(t) = e^{\lambda_j(t-t_0)}y_{0_j} + \int_{t_0}^t e^{\lambda_j(t-s)}h_j(y(s),s)ds$. We will now show that given $\varepsilon > 0$ there exist T_0, T such that for all $t_0 \geq T_0$, $t \geq T$, $|y_j(t)| < \varepsilon$.

We first consider $\left|\int_{t_0}^t e^{\lambda_j(t-s)}h_j(y(s),s)ds\right|$. We take λ_j to be real, and therefore, h_j is a real-valued function. We will later show that the case when λ_j is complex can be reduced to the real case. Taking λ_j to be real, we have $e^{\lambda_j(t-s)} > 0$. By the Mean Value Theorem for Integrals there exists $z \in [t_0, t]$ such that

$$\int_{t_0}^t e^{\lambda_j(t-s)}h_j(y(s),s)ds = h_j(y(z),z)\int_{t_0}^t e^{\lambda_j(t-s)}ds.$$

But

$$\int_{t_0}^t e^{\lambda_j(t-s)}ds = -\frac{1}{\lambda_j}\left(1 - e^{\lambda_j(t-t_0)}\right).$$

Since $\lambda_j < 0$ (ie: $-\lambda_j > 0$) and $t > t_0$, we have $\left|-\frac{1}{\lambda_j}\left(1 - e^{\lambda_j(t-t_0)}\right)\right| < -\frac{1}{\lambda_j}$. Therefore,

$$\left|\int_{t_0}^{t} e^{\lambda_j(t-s)} h_j(y(s),s)ds\right| < -\frac{1}{\lambda_j}\left|h_j(y(z),z)\right|.$$

Since $G(y,t)$ converges to 0 uniformly in y as t approaches infinty, h_j inherits this condition. Hence, since $z > t_0$, for any $\varepsilon > 0$, there exists T_0 such that for all $t_0 \geq T_0$, $|h_j(y(z),z)| < \frac{(-\lambda_j)\varepsilon}{2}$. Therefore, for $t_0 \geq T_0$, $\left|\int_{t_0}^{t} e^{\lambda_j(t-s)} h_j(y(s),s)ds\right| < -\frac{1}{\lambda_j} \cdot \frac{(-\lambda_j)\varepsilon}{2} = \varepsilon/2$.

In the case of complex eigenvalues, λ_j can be written as $\lambda_j = \rho_j + i\theta_j$, where $\rho_j < 0$ and θ_j are both real. Similarly, we can write $h_j(y(s),s) = u(s) + iv(s)$ where u and v are both real-valued functions converging to 0 uniformly in y as s approaches infinity. Then

$$\begin{aligned}\int_{t_0}^{t} e^{\lambda_j(t-s)} h_j(y(s),s)ds &= \int_{t_0}^{t} e^{\rho_j(t-s)}(u(s)\cos(\theta(t-s)) - v(s)\sin(\theta(t-s)))ds\ldots \\ &+ i\int_{t_0}^{t} e^{\rho_j(t-s)}(u(s)\sin(\theta(t-s)) + v(s)\cos(\theta(t-s)))ds.\end{aligned}$$

Hence, $\left|\int_{t_0}^{t} e^{\lambda_j(t-s)} h_j(y(s),s)ds\right| \leq 2\left|\int_{t_0}^{t} e^{\rho_j(t-s)}(u(s)+v(s))\right|$, thereby reducing the complex case to the real case.

We now consider $\left|e^{\lambda_j(t-t_0)} y_{0_j}\right|$; $|y_{0_j}|$ is a constant, which we will denote C_j. Since λ_j has negative real part, there exists T_1 such that for all $t - t_0 \geq T_1$, $\left|e^{\lambda_j(t-t_0)} y_{0_j}\right| = C\left|e^{\lambda_j(t-t_0)}\right| < C \cdot \frac{\varepsilon}{2c} = \frac{\varepsilon}{2}$. Therefore, we have shown that for $T = t_0 + T_1$, for all

$t_0 \geq T_0$, and $t \geq T$,

$$|y_j(t)| \leq |e^{\lambda_j(t-t_0)} y_{0_j}| + \left| \int_{t_0}^{t} e^{\lambda_j(t-s)} h_j(y(s), s) ds \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies the convergence of $\|y(t)\|$ to 0, and therefore, the solution $x(t)$ of $\frac{dx}{dt} = Ax + G(x,t)$ to 0 as well. □

Before we apply this lemma to our system, let us just take a moment to recall our other assumptions: $0 < a < 2b$, $0 < a + c < 2b$, $0 < d < 2e$, $0 < d + f < 2e$, $cf \neq 4eb$ and our Nash equilibrium must lie in $[0, 1]$. Consider the matrix A corresponding to the system of equations given in (5) taken in the limit:

$$A = \begin{pmatrix} -1 & \frac{c}{2b} \\ \frac{f}{2e} & -1 \end{pmatrix}. \tag{6}$$

The eigenvalues for A are

$$l_1, l_2 = \left(-1 \pm \sqrt{\frac{cf}{4be}} \right).$$

Under our given assumptions, this yields only certain possibilities for our eigenvalues:

Lemma 10 *Under the given conditions for the coefficients, the eigenvalues of A are either both real distinct and negative, both complex with negative real parts, or both the same nonpositive number.*

Proof: To run through all the cases, we first consider the case where we have real distinct eigenvalues. In order for our eigenvalues to be real, we must have $\frac{cf}{be} > 0$. It is clear by looking at the formula for l_1 and l_2 that they will never both be positive. Since $be > 0$, this means we must have $cf > 0$ too, so either $c, f > 0$ or $c, f < 0$. It is clear from the formula for l_1, l_2 that l_1 and l_2 will both be negative if $4be > cf$ or of different signs if $4be < cf$. It is possible for l_1 and l_2 to both be negative, and an example of this is given in the equations of Figure 1 in Section 6. Slightly less apparent is the fact that we cannot have eigenvalues of opposite signs. We first show that to have this case, we must have $c, f > 0$:

$$cf > 4be = (2b)(2e) > (a + c)(d + f) = ad + dc + fa + cf.$$

Hence, $0 > ad + dc + cf$. Therefore, $c, f < 0$ since $a, d > 0$. By the bounds on our Nash equilibrium, we know $0 < cd + 2ea < 4eb - cf$. Hence $-cd < 2ea$. But $a < 2b$, so $-cd < 2ea < 4eb$. Hence $-cd < cf$ since $4eb < cf$. Therefore, $-d > f$ since $c < 0$. Hence $0 > d + f$, a contradiction. Therefore, in order for the eigenvalues of A to be real and distinct, they must be negative.

The next possiblity is for the eigenvalues to be complex conjugates. It is clear from the formula that they will always have a real part of -1. These kinds of eigenvalues are attainable, and examples of them can be found in the equations to Figures 2 and 3 in Section 6.

Finally, the eigenvalues may be described by the case when $\frac{cf}{eb} = 0$, or when $l_1 = l_2$. This case is attainable, but only occurs when $c = 0$ or $f = 0$. This case does not qualify as a game, so we ignore this case in further discussion. $\square$

We now return to Lemma 9 to establish convergence of the peaks ξ and η to the unique interior Nash equilibrium.

Theorem 3 *In the two-population linear quadratic case, the conditions* $c, f \neq 0$ *and* $0 \leq \xi_N, \eta_N \leq 1$*, together with conditions necessary to apply the Peak Behavior Theorem are enough to imply the stability of the solutions to the equations (3), and hence, the convergence of the peaks* ξ *and* η *to the unique interior Nash equilibrium.*

Proof: Taking A as described above in equation (6), $x = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$, where ξ' and η' are as described for equation (5), and

$$G(x,s) = \begin{pmatrix} \frac{2b\xi' - c(\eta' + \phi_y(e^s))}{e^{-s}K_\mu(\eta' + \eta_N) - 2b} \\ \frac{2e\eta' - f(\xi' + \phi_y(e^s))}{e^{-s}K_\lambda(\xi' + \xi_N) - 2e}. \end{pmatrix} - Ax,$$

we seek to apply Lemma 9. $G(x,s)$ converges to 0 uniformly in x as s approaches infinity, and G is continuous and bounded for all x and s. Moreover, by Lemma 10, A has distinct eigenvalues, and is hence diagonalizable. Therefore, the solution $x = 0$ to the equations described in (5) is asymptotically stable Lemma 9. Therefore, we can conclude that the solution $\xi(t) = \xi_N$, $\eta(t) = \eta_N$ is an asymptotically stable solution to (3). Hence, the peaks ξ and η converge to the unique interior Nash equilibrium.

□

Numerical Simulations

We now wish to observe our theoretical results by examining some simulations. The MATLAB code evolver.m found in the appendix provides the foundation for our numerical treatment. The code takes a payoff function P together with a second payoff function Q and evolves uniform initial population densities under the replicator dynamics using Euler's method.

We first initialize the time step to 0.5, and then we discretize the continuum of decisions into 100 pure decision choices. The population densities are set to a uniform density. Next, the figure is configured to display multiple plots. In each plot, the dotted line corresponds to the μ population of players with payoff function P, and the solid line corresponds to the λ population of players with payoff function Q. The program then sets up the payoff structures using the payoff functions.

At this point, the program applies Euler's method. For 10 iterations, it runs $25i$ steps of time in the ith iteration, applying Euler's method to the two population replicator dynamics

$$\frac{\partial \mu}{\partial t} = \mu_t(x)(P^*(x|\lambda_t) - F(\mu_t|\lambda_t)) \quad \text{and} \quad \frac{\partial \lambda}{\partial t} = \lambda_t(y)(Q^*(y|\mu_t) - G(\lambda_t|\mu_t)),$$

and producing a plot after each iteration. The densities evolve using the recursive formulas $\mu_i = \mu_{i-1} + h(P^*(x|\mu_{i-1}) - F(\mu_{i-1}|\lambda_{i-1}))$ and $\lambda_i = \lambda_{i-1} + h(Q^*(y|\lambda_{i-1}) - G(\mu_i|\lambda_{i-1}))$. After each application, we ensure that our densities remain nonnegative by resetting all negative values in μ and λ to 0. Finally, since it is easier to do calculations using the interval $[1, 100]$ broken into integers, we normalize our density function after every step to be in the $[0, 1]$ interval.

We apply our code to functions meeting the criteria we discussed for both stable and nonstable Nash equilibria. Here, as before, ξ_N refers to the Nash equilibrium of the μ population, and η_N refers to the Nash equilibrium of the λ population.

We first examine the theoretically guaranteed stable cases. Two payoff functions that will yield real distinct negative eigenvalues are $P(x,y) = x(1-2x+2y)$, which is given in pstable.m, and $Q(x,y) = y(1-2y+x)$, given in qstable.m. As computed using Lemma 8, these equations have a Nash equilibrium of appproximately $\xi_N = 0.4286$ and $\eta_N = 0.3571$. As we can see from Figure 1, both populations are converging to support concentrated at this Nash equilibrium.

This brings us to the final guaranteed case of stability where the matrix corresponding to our equations has complex eigenvalues with negative real parts. The first example is given by ppayoff2.m, $P(x,y) = x(0.5 - x - y)$, and qpayoff2.m, $Q(x,y) = y(0.5 - y + x)$. This has a Nash equilibrium of $\xi_N = 0.1$ and $\eta_N = 0.3$. A quick check to Figure 2 shows that the respective populations are nearly centered at their respective Nash equilibria. A second example of this case is given by pspiral2.m, $P(x,y) = x(1 - 2x + y)$, and qspiral2.m, $Q(x,y) = y(3 - 3y - 2x)$. This example has a computed Nash equilibrium of $\xi_N = \frac{9}{26}$ and $\eta_N = \frac{10}{26}$ by Lemma 8. Refer to Figure 3 to check the example against our theoretical predictions.

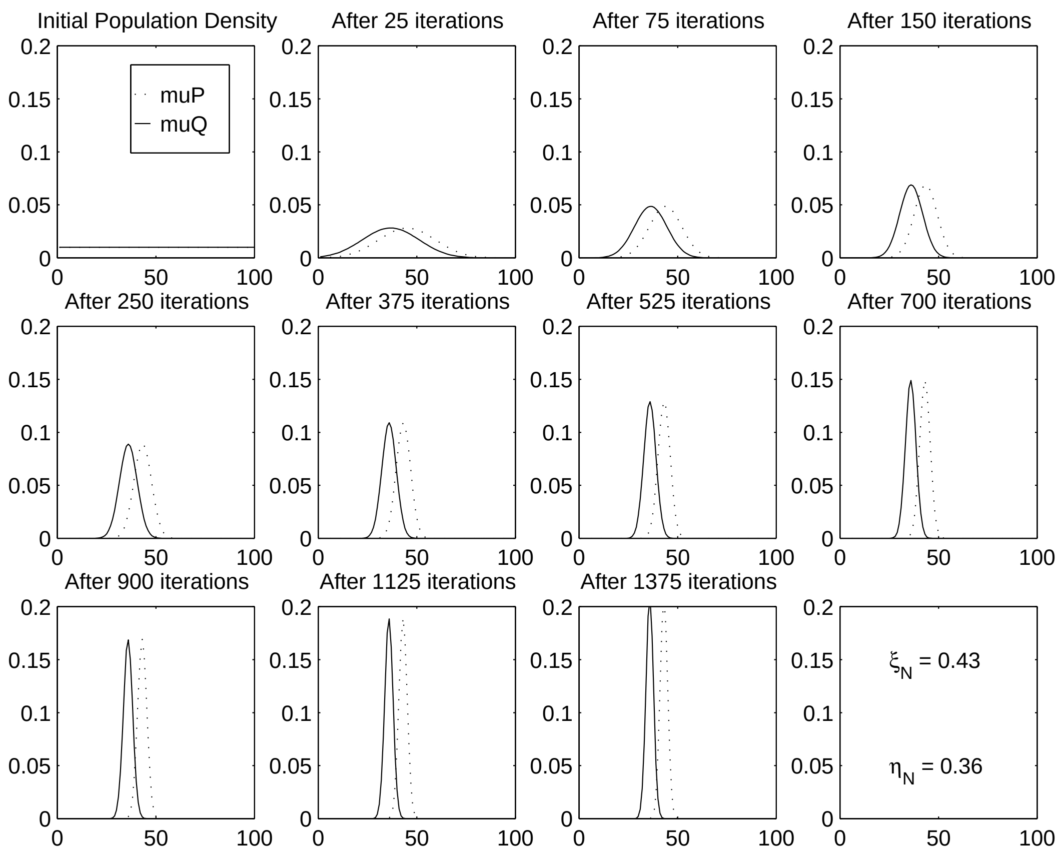

Figure 1: Evolution of $P(x, y) = x(1 - 2x + 2y)$ and $Q(x, y) = y(1 - 2y + x)$

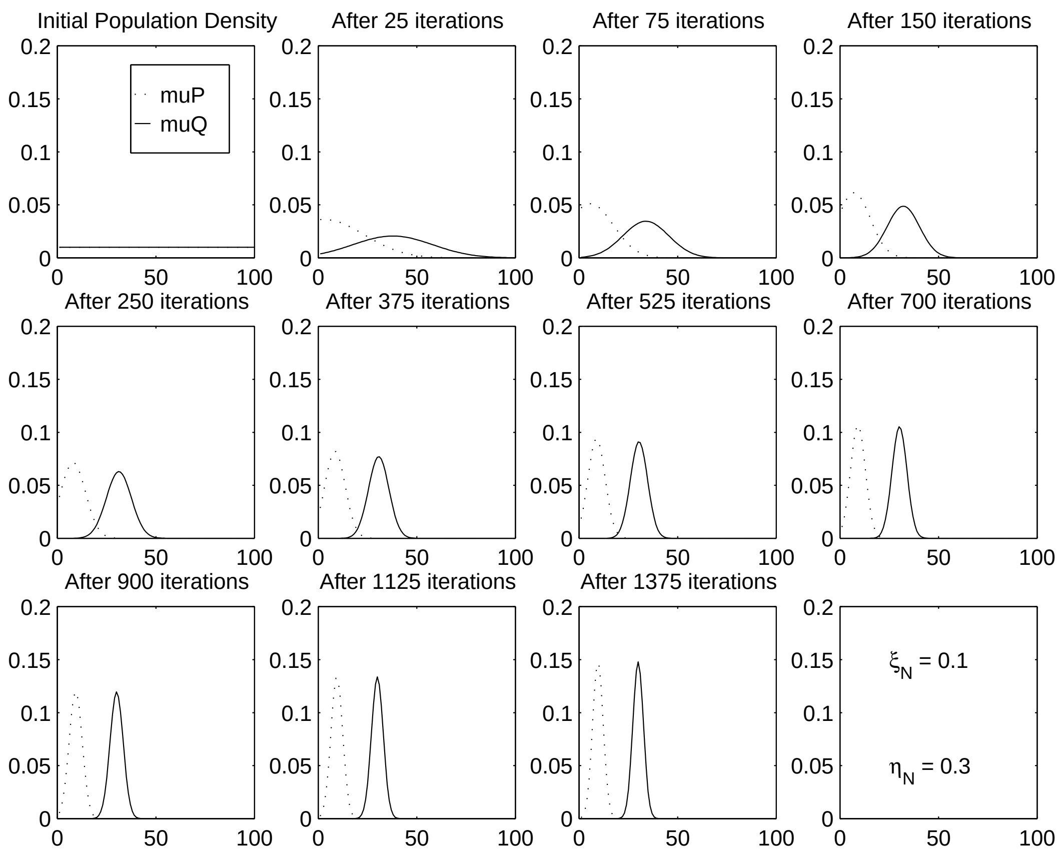

Figure 2: Evolution of $P(x, y) = x(0.5 - x - y)$ and $Q(x, y) = y(0.5 - y + x)$

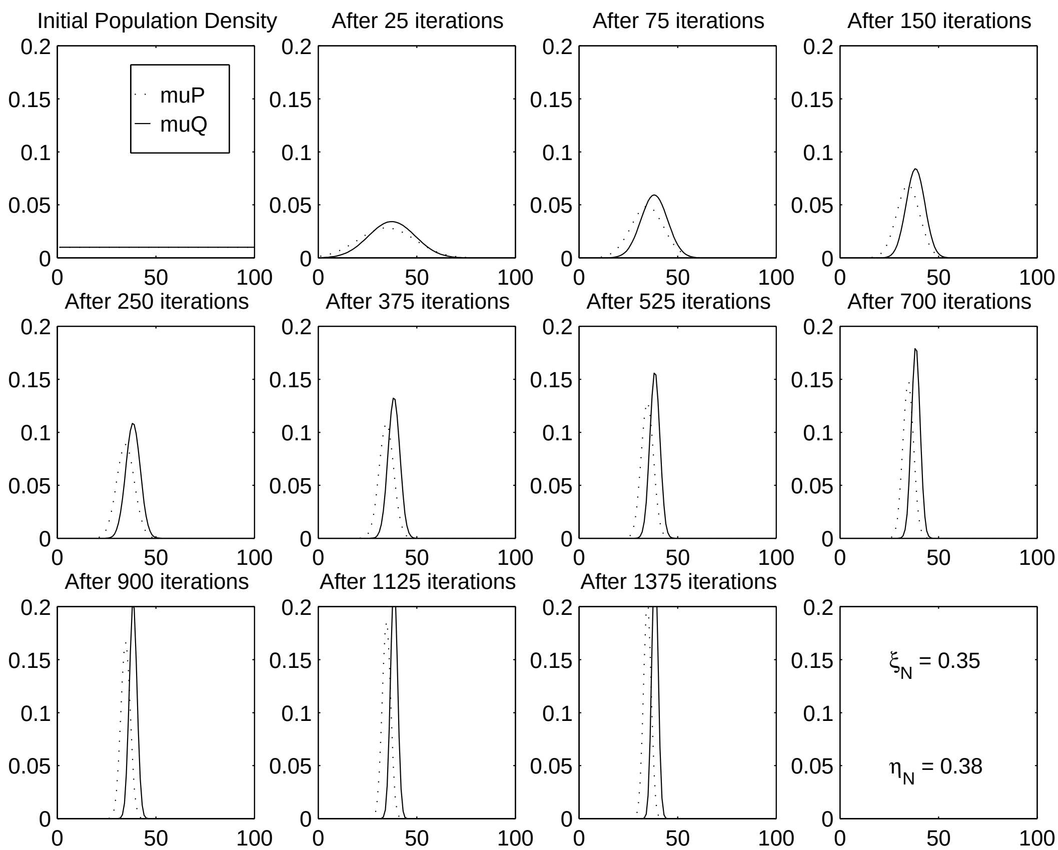

Figure 3: Evolution of $P(x,y) = x(1-2x+y)$ and $Q(x,y) = y(3-3y-2x)$

Finally, when we relax the conditions on the first partial derivatives of P and Q needed to prove the Peak Behavior Theorem, we can produce an unstable interior Nash equilibrium. This example has symmetric payoffs and is given by punstable.m, $P(x, y) = x(1-x-3y)$, and qunstable.m, $Q(x, y) = y(1-y-3x)$. Rather than evolving toward the interior Nash equilibrium computed in Lemma 8 to be $\eta_N = \xi_N = 0.2$, our densities now evolve toward a Nash equilibrium on the boundary. This can be seen in Figure 4 as the maxizands move away from each other.

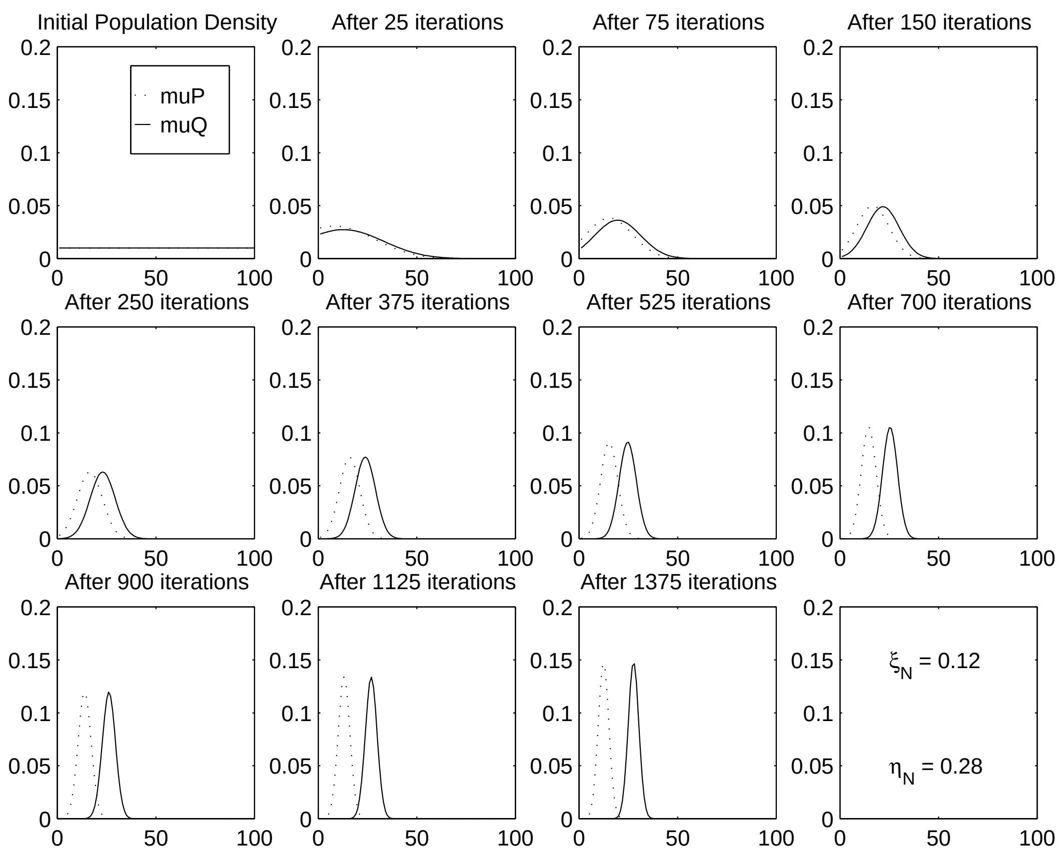

Figure 4: Evolution of $P(x, y) = x(1 - x - 3y)$ and $Q(x, y) = y(1 - y - 3x)$

Discussion

In this paper, we have examined the evolution of two-populations with linear quadratic payoff functions under the replicator dynamics. Given payoff functions concave in the decision maker's variable and given boundary conditions ensuring our dynamics remains in the interior of $(0,1) \times (0,1)$, in the Two-Population Mass Concentration Theorem and the Peak Behavior rm, we have provided the natural extension of results found in [5]. Namely, we have found that in the two-population replicator dynamics, the populations will converge to unique peaks.

We then examined populations taking linear quadratic payoff functions, $P(x,y) = x(a - bx + cy)$ and $Q(x,y) = y(d - ey + fx)$, that determine a game (ie: $cf \neq 0$). Using a result that allowed us to examine the stability peak behavior in the limit, we have been able to further extend the results in [5] to show that the populations will converge to unique interior peaks at the Nash equilibrium.

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Matlab Code

evolver.m

```
% EVOLVER
%
%    EVOLVER(ppayofffunction, qpayofffunction) takes p player's
%    payoff function and q player's payoff function and applies
%    the replicator dynamics to them
%

function [] = evolver(ppayofffunction, qpayofffunction)

n=100;
h=0.5;

% Initialize our population densities
% This is a uniform density
for k = 1:n,
   mup(k) = 1/n;   % mup = The population density of P players
   muq(k) = 1/n;   % muq = the population density of Q players
end

x=1:100;   % x contains the x-coordinate for our graphs

figure(1);
axiswindow = [0, 100, 0, 0.2];
%subplot(3,4,1), plot(x, mup, 'b^', x, muq, 'rv');
subplot(3,4,1), plot(x, mup, 'b:', x, muq, 'r');
axis(axiswindow);   % Uniform axes for the plots for comparison
title(['Initial Population Density']);
legend('muP','muQ');

% Now we set up our payoff structures
for a=1:n,
   for b=1:n,

      % Ptable is the matrix that lists the payoff for a p player
```

```
      % when playing a against b.  We divide by n to get our payoffs
      % in the [0,1] interval.
      PTable(a,b) = feval(ppayofffunction,a./n, b./n);

      % Qtable is similarly a table that lists the payoff for
      % a q player when playing b against a
      QTable(a,b) = feval(qpayofffunction,a./n, b./n);
   end
end


numberiterations = 0;

for s=1:1
   figure(s);

for l=1:10,  % After time passes, we create another plot.
   for k=1:(25*l),   % This loop passes time.

      % Setting up the fitness table of a p player
      for a=1:n,

         % PTable(a,:) gives only the entries from row a...  It is a
         % row.  muq is also a row.  FitnessP is also a row.
         FitnessP(a) = sum(PTable(a,:) .* muq);
      end

      % is a row vector
      FtP = sum(FitnessP .* mup) * ones(1,n);

      % is a row vector
      FDP = FitnessP - FtP;

      mup = mup .* (ones(size(FDP)) + h*FDP);

      for a=1:n
         if mup(a)<=0
            mup(a) = 0;
```

```
      end
   end

   % must "normalize" mup
   S = sum(mup);
   mup = mup / S;

   % Setting up the fitness table of a q player
   for a=1:n,

      % QTable(:,a) gives only the entries from column a...
      % It is a column.  muq is also a row.  FitnessP is
      % also a row.
      FitnessQ(a) = sum(QTable(:,a)' .* mup);
   end

   % is a row vector
   FtQ = sum(FitnessQ .* muq) * ones(1,n);

   % is a row vector
   FDQ = FitnessQ - FtQ;

   muq = muq .* (ones(size(FDQ)) + h*FDQ);

   for a=1:n
      if muq(a)<=0
         muq(a) = 0;
      end
   end

   % must "normalize" mup
   S = sum(muq);
   muq = muq / S;

end

subplot(3,4,l+1), plot(x, mup, 'b:', x, muq, 'r');
axis(axiswindow);
```

```
   numberiterations = k+numberiterations;
   title(['After ', int2str(numberiterations), ' iterations']);
end

[maximummup,xi]=max(mup);
[maximummupq,eta]=max(muq);

xi = xi/100;
eta = eta/100;

subplot(3,4,12), plot([1,1], [2,1],'o');
axis(axiswindow);
text(25,.15, ['\xi_N = ', num2str(xi)]);
text(25,.05, ['\eta_N = ', num2str(eta)]);
sprintf('After %d iterations, xi = %0.5g and eta =
%0.5g',numberiterations,xi,eta)

end
```

ppayoff2.m

```
% [payoff] = ppayoff2(x,y).
% This is the concave payoff function P(x,y)=x .*(0.5-x-y)

function [payoff] = ppayoff2(x,y)
payoff = x .* (0.5-x-y);
```

pspiral2.m

```
% [payoff] = pspiral2(x,y).
% This is the concave payoff function P(x,y)=x .*(1-2x+y)

function [payoff] = pspiral2(x,y)
payoff = x .* (1-2*x+y);
```

pstable.m

```
% [payoff] = pstable(x,y).
% This is the concave payoff function P(x,y)=x .*(1-2x+2y)

function [payoff] = pstable(x,y)
payoff = x .* (1-2*x+2*y);
```

punstable.m

```
% [payoff] = punstable(x,y).
% This is the concave payoff function P(x,y)=x .*(1-x-3y)

function [payoff] = punstable(x,y)
payoff = x .* (1-x-3*y);
```

qpayoff2.m

```
% [payoff] = qpayoff2(x,y).
% This is the concave payoff function Q(x,y)=y .*(0.5+x-y)

function [payoff] = qpayoff2(x,y)
payoff = y .* (0.5+x-y);
```

qspiral2.m

```
% [payoff] = qspiral2(x,y).
% This is the concave payoff function Q(x,y)=y .*(3-3y-2x)

function [payoff] = qspiral2(x,y)
payoff = y .* (3-3*y-2*x);
```

qstable.m

```
% [payoff] = qstable(x,y).
% This is the concave payoff function Q(x,y)=y .*(1-2y+1x)

function [payoff] = qstable(x,y)
payoff = y .* (1-2*y+x);
```

qunstable.m

```
% [payoff] = qunstable(x,y).
% This is the concave payoff function Q(x,y)=y .*(1-y-3*x)

function [payoff] = qunstable(x,y)
payoff = y .* (1-y-3*x);
```